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Short Communication

A modified eigenfunction expansion approximation for nonlinear oscillating equations

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Abstract

A modified eigenfunction expansion techniques is presented for constructing analytical approximate solutions of the nonlinear oscillatory equations. This method is valid for both small and large parameters and large oscillation amplitude. Two examples are given to illustrate the effectiveness of the proposed method.

There are various techniques for constructing analytical approximations to the oscillatory solutions of second-order nonlinear differential equations [1,2]. But many of them are applied to weakly nonlinear cases only and are valid for small parameters. To overcome the limitations, many novel methods have been proposed in recent years. For example, Cheung et al. [3] proposed a modified Lindstedt–Poincare method and Lim et al. [4] presented a modified Mikens procedure for certain nonlinear oscillators. Recently, He [5] proposed a perturbation technique and Hu [6] presented a classical perturbation technique. Both methods are valid for large parameters. Furthermore, the nonlinear eigenexpansion is also an effective tool in solving the nonlinear problems. This paper shall study a modified eigenfunction expansion techniques for constructing

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analytical approximate solutions of the nonlinear oscillatory equations. This method is valid for both small and large parameters and large oscillation amplitudes.

Consider a nonlinear oscillator modeled by the equation

$$\ddot{x} + f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denotes differentiation with respect to time t and $f(x)$ satisfies the condition

$$f(-x) = -f(x). \tag{2}$$

Let ω be the fundamental oscillatory frequency of Eq. (1) and introducing the substitution $\theta = \omega t$, $\mathrm{d}/\mathrm{d}t = \omega \mathrm{d}/\mathrm{d}\theta$ into Eq. (1), we obtain

$$\omega^2 x'' + f(x) = 0, \quad x(0) = A, \quad x'(0) = 0, \tag{3}$$

where primes designate differentiation with respect to θ. So the solution $x(\theta)$ of Eq. (3) is the periodic function of variable θ with period 2π.

The solution restricted in interval $[\pi/2, 3\pi/2]$ of Eq. (3) is equivalent to the minimization of the energy functional:

$$J(x) = \int_{\pi/2}^{3\pi/2} \left[\frac{1}{2}\omega^2 (x')^2 - F(x)\right] \mathrm{d}\theta, \tag{4}$$

where potential

$$F(x) = \int_0^x f(\xi)\, \mathrm{d}\xi. \tag{5}$$

For our discussion, consider the eigenvalue problem

$$\varphi_j''(\theta) + \lambda_j \varphi_j(\theta) = 0, \quad \varphi_j\left(\frac{\pi}{2}\right) = \varphi_j\left(\frac{3\pi}{2}\right) = 0 \tag{6}$$

and its eigenpairs $\lambda_j, \varphi_j(\theta)$ satisfy

$$\lambda_j = (2j-1)^2, \quad \varphi_j(\theta) = \cos\sqrt{\lambda_j}\theta, \quad j = 1, 2, 3, \ldots, \tag{7}$$

where the eigenfunctions $\{\varphi_j\}_1^\infty$ form a complete orthogonal system. We denote the finite dimensional space

$$V_n = \mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\},$$

then the nth order approximate periodic solution of Eq. (3) can be decided by the following Fourier series

$$x_n(\theta) = \sum_{j=1}^{n} a_j \varphi_j(\theta) = \sum_{j=1}^{n} a_j \cos\sqrt{\lambda_j}\theta \in V_n, \tag{8}$$

which is rewritten as the formula with respect to time t;

$$x_n(t) = \sum_{j=1}^{n} a_j \cos\sqrt{\lambda_j}\omega t. \tag{9}$$

To determine these coefficients $a_1, a_2, \ldots, a_n$, by inserting Eq. (8) into Eq. (4), we obtain

$$J(x_n) = \frac{\pi\omega^2}{4}\sum_{j=1}^{n}\lambda_j a_j^2 - \int_{\pi/2}^{3\pi/2} F\left(\sum_{j=1}^{n} a_j\varphi_j\right) \mathrm{d}\theta. \tag{10}$$

For a minimum to occur, considering J as a function of $a_1, a_2, \ldots, a_n$, it is necessary to have

$$\frac{\partial J}{\partial a_j} = 0 \quad \text{for each } i = 1, 2, \ldots, n. \tag{11}$$

Differentiating Eq. (10) gives

$$\begin{aligned}\frac{\partial J(x)}{\partial a_i} &= \frac{1}{2}\pi\lambda_i\omega^2 a_i - \int_{\pi/2}^{3\pi/2} F\left(\sum_{j=1}^{n} a_j\varphi_j\right)\varphi_i \,\mathrm{d}\theta \\ &= \frac{1}{2}\pi\lambda_i a_{n+1} a_i - \int_{\pi/2}^{3\pi/2} F\left(\sum_{j=1}^{n} a_j\varphi_j\right)\varphi_i \,\mathrm{d}\theta = 0, \quad i = 1, 2, 3, \ldots, n,\end{aligned} \tag{12}$$

where

$$a_{n+1} = \omega^2, \quad \omega = \sqrt{a_{n+1}}. \tag{13}$$

Recalling the initial condition of Eq. (1), we have another equation by Eq. (8)

$$a_1 + a_2 + a_3 + \cdots + a_n - A = 0. \tag{14}$$

Combination of Eqs. (12) and (14) is a nonlinear system of equations with respect to $a_1, a_2, \ldots, a_n, a_{n+1}$ which can be solved by Newtonian iterative method.

The nonlinear system of Eqs. (12) and (14) is rewritten in the vector form as follows:

$$\mathbf{F}(\mathbf{U}) = \mathbf{0}, \tag{15}$$

where

$$\mathbf{U} = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \\ a_{n+1} \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

and

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \lambda_1 a_{n+1} a_1 - \frac{\pi}{2}\int_{\pi/2}^{3\pi/2} f\left(\sum_{j=1}^{n} a_j \varphi_j\right)\varphi_1 \,\mathrm{d}\theta \\ \lambda_2 a_{n+1} a_2 - \frac{\pi}{2}\int_{\pi/2}^{3\pi/2} f\left(\sum_{j=1}^{n} a_j \varphi_j\right)\varphi_2 \,\mathrm{d}\theta \\ \cdots \\ \lambda_n a_{n+1} a_n - \frac{\pi}{2}\int_{\pi/2}^{3\pi/2} f\left(\sum_{j=1}^{n} a_j \varphi_j\right)\varphi_n \,\mathrm{d}\theta \\ a_1 + a_2 + a_3 + \cdots + a_n - A \end{pmatrix}. \tag{17}$$

The Jacobian matrix of $\mathbf{F}(\mathbf{U})$ is

$$\mathbf{DF}(\mathbf{U}) = \begin{pmatrix} \lambda_1 a_{n+1} - b_{11} & -b_{12} & \cdots & -b_{1n} & \lambda_1 a_1 \\ -b_{21} & \lambda_2 a_{n+1} - b_{22} & \cdots & -b_{2n} & \lambda_2 a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{n1} & -b_{n2} & \cdots & \lambda_n a_{n+1} - b_{nn} & \lambda_n a_n \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \tag{18}$$

where

$$b_{ij} = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} f'\left(\sum_{k=1}^{n} a_k \varphi_k(\theta)\right)\varphi_i(\theta)\varphi_j(\theta)\,\mathrm{d}\theta. \tag{19}$$

So the Newtonian iterative technique has the form

$$\mathbf{U}^{(k)} = \mathbf{U}^{(k-1)} - [\mathbf{DF}(\mathbf{U}^{(k-1)})]^{-1}\mathbf{F}(\mathbf{U}^{(k-1)}), \quad k = 1, 2, 3 \ldots. \tag{20}$$

With $n = 1$ in Eqs. (12) and (14), we get

$$a_1^{(0)} = A, \quad a_{n+1}^{(0)} = \frac{2}{\pi A}\int_{\pi/2}^{3\pi/2} f(A\varphi_1)\varphi_1 \,\mathrm{d}\theta. \tag{21}$$

It is known that one has to start the Newton iteration with an initial approximation $\mathbf{U}^{(0)}$ sufficiently close to $\mathbf{U}^{(0)}$, hence an initial approximation

$$\mathbf{U}^{(0)} = (a_1^{(0)}, 0, \ldots, 0, a_{n+1}^{(0)})^{\mathrm{T}} \tag{22}$$

for an initial value in the following iterative algorithm.

The Newton iterative algorithm to compute the nth order approximate solution of the nonlinear system $\mathbf{F}(\mathbf{U}) = 0$ is described by the following algorithm.

Algorithm 1. To obtain a solution to $\mathbf{F}(\mathbf{U}) = \mathbf{0}$ given an initial approximation $\mathbf{U}^{(0)}$ by Eq. (22):

INPUT: Number n of approximate order; initial approximation $\mathbf{U}^{(0)}$; tolerance TOL; maximum number of iterations N.
OUTPUT: Approximate solution $\mathbf{U}$ or a message that the number of iterations was exceeded.
Step 1: Set $k = 1$.
Step 2: While $k \leqslant N$ do Steps 3–8.
Step 3: Calculate $\mathbf{F}(\mathbf{U}^{(0)})$ by Eq. (17) and $\mathbf{DF}(\mathbf{U}^{(0)})$ by Eq. (18), whose elements can be computed by composite Gaussian quadrature.
Step 4: Solve the $(n+1)$-order linear system $\mathbf{DF}(\mathbf{U}^{(0)})\mathbf{V} = -\mathbf{F}(\mathbf{U}^{(0)})$.
Step 5: Calculate $\|\mathbf{V}\| = \max_{0 \leqslant j \leqslant n+1} |\mathbf{V}(j)|$.
Step 6: If $\|\mathbf{V}\| \leqslant$ TOL, then
set $\mathbf{U} = \mathbf{U}^{(0)} + \mathbf{V}$,
OUTPUT($\mathbf{U}$); (*The procedure was successful.*)
STOP.
Step 7: Set $k = k + 1$.
Step 8: Set $\mathbf{U}^{(0)} = \mathbf{U}$. (*Update* $\mathbf{U}^{(0)}$.)
Step 9: OUTPUT('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

From Algorithm 1 we can obtain approximate Fourier coefficients $a_j = \mathbf{U}(j), j = 1, 2, \ldots, n$ and approximate frequency $\omega = \sqrt{\mathbf{U}(n+1)}$.

In order to analyze the eigenfunction expansion approximation, we take the fractional power equations

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0 \tag{23}$$

for example. We can obtain three class of approximate solutions and frequencies of Eq. (23) as follows:

The third approximate solution to Eq. (23) is

$$x_3 = 1.019542 \cos \omega_3 t - 0.023800 \cos 3\omega_3 t + 0.004257 \cos 5\omega_3 t \tag{24}$$

with

$$\omega_3 = 1.070773.$$

The fifth approximate solution to Eq. (23) is

$$\begin{aligned} x_5 = {} & 1.020349 \cos \omega_5 t - 0.023835 \cos 3\omega_5 t + 0.000427 \cos 5\omega_5 t \\ & - 0.000138 \cos 7\omega_5 t + 0.000596 \cos 9\omega_5 t, \end{aligned} \tag{25}$$

with

$$\omega_5 = 1.070515.$$

The 10th approximate solution to Eq. (23) is

$$\begin{aligned} x_{10} = 1.020572 \cos \omega_{10}t - 0.023843 \cos 3\omega_{10}t + 0.004275 \cos 5\omega_{10}t \\ - 0.001384 \cos 7\omega_{10}t + 0.000597 \cos 9\omega_{10}t) - 0.000305 \cos 11\omega_{10}t \\ + 0.000174 \cos 13\omega_{10}t - 0.000108 \cos 15\omega_{10}t + 0.000071 \cos 17\omega_{10}t \\ - 0.000049 \cos 19\omega_{10}t, \end{aligned} \tag{26}$$

with

$$\omega_{10} = 1.070443.$$

The exact frequency of Eq. (23) is given by [7]

$$\omega_{1/3}^{\mathrm{EX}} = \frac{\sqrt{\pi}\Gamma(1/4)}{2\sqrt{6}\Gamma(3/4)} = 1.070451, \tag{27}$$

then the relative error of the third, fifth and 10th approximate frequency with respect to the exact frequency are 2.63/10 000, 5.98/100 000 and 7.86/1 000 000, respectively.

We also consider the Duffing equation

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \tag{28}$$

for example where ε is a positive parameter. To compare the present results with exact results, we take $\omega_0^2 = 1$. The exact frequency of the periodic motion of the Duffing equation is given by [8]

$$\omega_e = \frac{\pi\sqrt{1+\varepsilon A^2}}{2}\left(\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)}. \tag{29}$$

Table 1
Comparison of approximate frequency with the exact frequency for the Duffing equation

εA^2	ω_e	ω_3 (Relative error)	ω_5 (Relative error)
0.00001	1.00000037	1.00000037(0)	1.00000037(0)
0.0001	1.00000374	1.00000374(0)	1.00000374(0)
0.001	1.0000375	1.0000375(2.2204e-16)	1.0000375(2.2204e-16)
0.01	1.0037418	1.0037418(2.2874e-12)	1.0037418 (1.3313e-12)
0.1	1.0367169	1.0367169(1.8864e-9)	1.0367169 (1.0315e-10)
1	1.3177761	1.3177794()2.5226e-6	1.3177761(1.1092e-8)
10	2.8666403	2.8667866(5.1016e-5)	2.8666404 (1.3482e-8)
100	8.5335862	8.5342957(8.3146e-5)	8.5335876(1.5782e-7)
1000	26.810738	26.813088(8.7637e-5)	26.810743(1.8298e-7)
10 000	84.727479	84.734944(8.8103e-5)	84.727494(1.8565e-7)
100 000	267.91425	267.93787(8.8150e-5)	267.91430(1.8592e-7)
1 000 000	847.21369	847.28838(8.8154e-5)	847.21385(1.8594e-7)
10 000 000	2679.1232	2679.3594(8.8155e-5)	2679.1237(1.8594e-7)
100 000 000	8472.1308	8472.8777(8.8155e-5)	8472.1324(1.8595e-7)
1 000 000 000	26791.230	26793.592(8.8155e-5)	26791.235(1.8595e-7)

For comparison, the exact frequency ω_e obtained by integrating Eq. (29) and the third-order approximate frequency ω_3 and the fifth-order approximate frequency ω_5 computed by the modified eigenfunction expansion method (MEEM) are listed in Table 1.

The modified eigenfunction expansion techniques studied in this paper has been used to solve the Duffing equation. Table 1 indicates that this method can give excellent approximate frequencies for both small and large parameters and oscillation amplitude. For any values of εA^2, the maximal relative error of the third and the fifth approximate frequencies with respect to the exact solution is, respectively, less than 1/10 000 and 1/1 000 000 and convergent to zero as $\varepsilon A^2 \to 0$.

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